Gauge Kinematics of Deformable Bodies

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ABSTRACT

The treatment of the motion of deformable bodies requires a specification of axes for each shape. We present a natural kinematic formulation of this problem in terms of a gauge structure over the space of shapes that the body may assume. As an example, we discuss how deformations of a body with angular momentum zero can result in a change in orientation.

1. Introduction

Gauge potentials figure prominently in the formulation of fundamental physical laws. The abstractness of these laws, however, does not easily lend itself to an intuitive understanding of the concepts involved. Here we argue that gauge potentials arise naturally in a much more mundane, but in return more readily visualized, context—the description of the motion of deformable bodies. We hope that our exposition will provide both an introduction to some of the basic concepts of gauge theories and a useful framework for discussing the kinematics of deformable bodies.

A cat, held upside-down by its feet and released at rest from a suitable height, will almost always manage to land on its feet [1] [2]. A diver leaving the board with no angular momentum may perform several twists and somersaults before hitting the water [3] [4]. In both cases, by executing a sequence of deformations beginning and ending at the same shape, a deformable body with nothing to push against and no angular momentum has undergone a net rotation.

In this note, we will present a convenient and natural context for computing the net rotation of a body, in the absence of external forces and torques, due to a given sequence of deformations. Our starting point is the observation that such rotations have no dependence on the rate at which the deformations are made—the equations governing the motion are invariant under time reparameterizations. Only the geometry of the sequence of deformations matters. We shall show that the rotation of a self-deforming body may be naturally expressed in a purely geometric form, in terms of a gauge potential over configuration space.

A similar kinematic framework was devised recently for the description of another problem involving deformable bodies: swimming at low Reynolds number [5]. In that case, calculation of the gauge potential required the solution of a highly non-trivial hydrodynamic problem, while here we shall be able to write the complete solution in a simple, closed form.

The configuration space of a deformable body is the space of all possible shapes [5]. We should at the outset distinguish between the space of shapes located somewhere in space and the more abstract space of *unlocated* shapes. The latter space may be obtained from the space of shapes *cum* locations by declaring two shapes with different centers-of-mass and orientations to be equivalent. When no external forces act upon a deformable body, then we may always work in its center-of-mass frame, in which case the space of located shapes is just the space of shapes *with orientation* and centered at the origin.

The problem we wish to solve may be stated as follows: what is the net rotation which results when a deformable body goes through a given sequence of unoriented shapes, in the absence of external forces? In other words, given a path in the space of unlocated shapes, what is the corresponding path in the space of located shapes? The problem is intuitively well-defined—if a body changes its shape in some way, a net rotation is induced. This net rotation may be computed by making use of the law of conservation of angular momentum. Thus, if the body begins with some angular momentum L, then it will adjust its orientation in such a way as to preserve L. In general, this constraint is enough to determine fully the net rotation of the body.

These remarks may seem straightforward enough, but if we attempt to formulate them mathematically, we immediately run into a crucial ambiguity. Namely, how can we specify the net rotation of an object which is continuously changing its shape? The situation is illustrated in Fig. 1—in order to talk about the relative orientations of two shapes, we must choose a set of body-fixed axes for each. It would seem that for the problem at hand, there is a natural choice of axes for an arbitrary shape—its three moments of inertia. But even this choice is ambiguous: we must still specify which moments correspond to body-fixed $x-$, $y-$, and $z-$axes. One could then say that the x−axis is always the longest and the z−axis the shortest, but this choice becomes singular when two or more moments become degenerate, and the choice of axes for the sphere remains completely ambiguous.

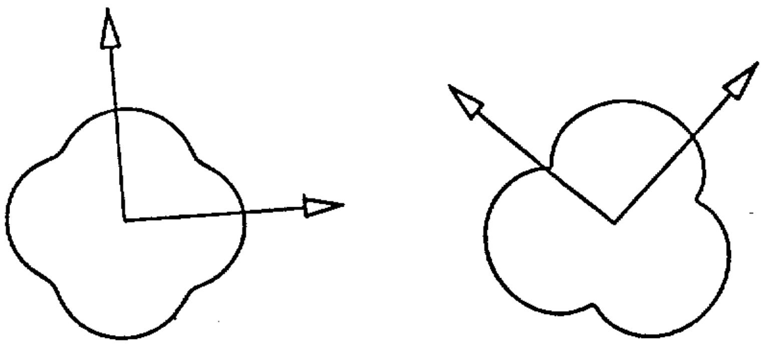

Figure 1. In order to measure the relative orientation of two different shapes, a choice of axes for each shape must be made.

Of course, we should not be too disoriented by this ambiguity, since whatever choice of axes we make, the problem still must have a solution. The distinction between particular choices is really no more than a matter of convenience.

We are faced with an enormous degeneracy of possible kinematic descriptions: at each point in an infinite-dimensional shape space, we must pick a set of reference axes from a space which looks like $SO(3)$. In the following section, we shall develop a formalism which works for any choice of axes, and which makes it easy to translate between different choices. The formalism is based on a *gauge structure* over shape space, and involves a construction known as a non-Abelian gauge potential. Although physicists first used gauge structures in the context of elementary particle theory, they have been shown to arise naturally in many other areas of physics and mathematics [5-9] . The present problem provides a further illustration of this universal concept.

2. Kinematics

Let us suppose that we have chosen a set of standard body-fixed references axes for each possible unoriented shape. Then every unoriented shape is associated to a standard oriented shape, whose $x-$, $y-$, and $z-$axes coincide with our choice of standard reference axes for the shape. For a given sequence of these standard shapes $S_0(t)$, we wish to find the corresponding sequence of physically oriented shapes $S(t)$, which are related by rotations $R(t)$:

$$S(t) = R(t)\, S_0(t) \tag{2.1}$$

$R(t)$ is a 3×3 rotation matrix which depends, in general, on the choice of reference axes for S_0. Indeed, if we make a local change in our standard shapes

$$\widetilde{S}_0 = \Omega[S_0]\, S_0 \tag{2.2}$$

then the physical shapes $S(t)$ must be unchanged, so

$$\widetilde{R}(t) = R(t)\, \Omega^{-1}[S_0(t)] \tag{2.3}$$

We shall compute $R(t)$ infinitesimally, and integrate to get the net rotation at finite times. We define

$$\frac{dR}{dt} = R\left[R^{-1}\frac{dR}{dt}\right] \equiv RA \tag{2.4}$$

A gives the infinitesimal rotation which results from the infinitesimal deformation of $S_0(t)$. We shall see presently that A is uniquely determined by the shape change. Once A is known, the full rotation at time t may be expressed as a path-ordered exponential

$$\begin{aligned} R(t) &= \overline{\mathrm{P}}\, \exp \int_0^t A(t')\, dt' \\ &\equiv \mathbf{1} + \int_{0<t'<t} A(t')\, dt' + \int_{0<t'<t''<t}\!\!\int A(t')A(t'')\, dt'dt'' + \cdots \end{aligned} \tag{2.5}$$

where the $\overline{\mathrm{P}}$ indicates that in expanding the exponential integral, all matrices are to be ordered with later times on the right. (For simplicity, we have taken $R(0) = \mathbf{1}$.)

The expression (2.5) is actually invariant under arbitrary time rescalings. Under $t \to \tau(t)$, the measure scales as $dt \to \dot{\tau}dt$, while $A \to A/\dot{\tau}$ since A contains one time derivative. This suggests that we should be able to write Eq. (2.5) in a completely geometric (*i.e.*, time-independent) form. In fact, we can define a *gauge potential* (or *connection*, to mathematicians) over the space of standard shapes S_0, which we shall also denote as A, by

$$A_{\dot{S}_0}[S_0(t)] \equiv A(t) \tag{2.6}$$

This definition requires some explanation. A is defined on the tangent space to S_0—that is, it is a vector field with a component for every direction of shape space—and it takes its values in the Lie algebra of infinitesimal $SO(3)$ rotations. In Eq.(2.6), A is evaluated at a particular shape $S_0(t)$, in the direction $\dot{S}_0$ in which the shape is changing. Thus, for a given infinitesimal deformation of S_0 by δS, the resulting net rotation of the shape is $A_{\delta S}[S_0]$. In terms of a fixed basis of tangent vectors $\{\omega_i\}$ at S_0, we can define components $A_i[S_0] \equiv A_{\omega_i}[S_0]$.

Now for a given path in shape space, the integration in (2.5) may be performed without referring to a time coordinate:

$$R(t) = \overline{\mathrm{P}} \exp \int^{S_0(t)} A[S_0] \cdot dS_0 \tag{2.7}$$

Multiplying this equation on the right by $\Omega^{-1}[S_0(t)]$ and differentiating in the ω_i direction shows that under the "gauge" transformation (2.2), (2.3), A transforms as a non-Abelian gauge potential should:

$$A_i \to \Omega A_i \Omega^{-1} + \Omega \nabla_i \Omega^{-1} \tag{2.8}$$

If one is only interested in rotations resulting from cyclic infinitesimal deformations of a shape S_0, then an approximate evaluation of the path-ordered exponential in Eq. (2.7) is possible. In the expansion of Eq. (2.5), each successive term will be down by a power of ϵ, where ϵ characterizes the size of the deformations. The first order term will vanish for a closed cycle, and the second order term may be written as an expression quadratic in A and linear in the first derivatives of A. It is this term we shall now compute.

Let the standard shapes near S_0 be parametrized by

$$S_0(t) = S_0 + s(t) \tag{2.9}$$

where the $s(t)$ are infinitesimal, of order ϵ. We expand $s(t)$ in terms of a basis of tangent vectors at S_0:

$$s(t) = \sum_i \alpha_i(t)\, \omega_i \tag{2.10}$$

Then the velocity in shape space is:

$$\dot{S}_0(t) = \sum_i \dot{\alpha}_i\, \omega_i \tag{2.11}$$

Now let us expand the gauge potentials to second order:

$$\begin{aligned} A_{\dot{S}_0}[S_0 + s(t)] &\cong A_{\dot{S}_0}[S_0] + \sum_i \frac{\partial A_{\dot{S}_0}}{\partial \omega_i} \alpha_i \\ &\cong \sum_j \left(A_j \dot{\alpha}_j + \sum_i \frac{\partial A_j}{\partial \omega_i} \alpha_i \dot{\alpha}_j \right) \end{aligned} \tag{2.12}$$

In the path ordered exponential integral (2.5) around a closed cycle, the first order term in (2.12) gives no contribution, for it is a total derivative. The second order contributions are terms quadratic in A and linear in its derivatives. Because (2.5) is gauge covariant for a cyclic path, its Taylor expansion in powers of $s(t)$ must also be gauge covariant, order by order. In fact, there is a unique (up to normalization) second order gauge covariant term we can form, which is antisymmetric in its indices i and j:

$$F_{ij} \equiv \frac{\partial A_i}{\partial \omega_j} - \frac{\partial A_j}{\partial \omega_i} + [A_i, A_j] \tag{2.13}$$

(Antisymmetry of F_{ij} means that the reverse cycle leads to the reverse rotation.) We shall call F the *field strength tensor* (or *curvature*) at S_0. It is easily verified that expansion of Eq. (2.5) to second order gives

$$\overline{\mathrm{P}} \exp \oint A\, dt = 1 + \tfrac{1}{2} \oint \sum_{ij} F_{ij}\, \alpha_i \dot{\alpha}_j\, dt \tag{2.14}$$

The field strength tensor, evaluated at a shape S_0, encodes all information on rotations due to arbitrary infinitesimal deformations of S_0.

3. Dynamics: Computing the Gauge Potential

The dynamics of a free self-deforming body is completely determined by the law of angular momentum conservation. The gauge potential A, in turn, completely describes the dynamics. In this section, we derive a general expression for A, for a body with angular momentum zero.

Let us consider a body which is a collection of point masses $m^{(n)}$ at $x^{(n)}$. Then the total angular momentum of the body is

$$L_i = \epsilon_{ijk} \sum_n m^{(n)}\, x_j^{(n)} \dot{x}_k^{(n)} \tag{3.1}$$

where the sums over repeated indices are implicit and all indices run from 1 to 3. Now to each possible configuration of the $x^{(n)}$'s is associated a unique standardly oriented configuration $\tilde{x}^{(n)}$. At time t, the two configurations are related by a rotation:

$$x^{(n)}(t) = R(t)\tilde{x}^{(n)}(t) \tag{3.2}$$

Thus, expressed in terms of $\tilde{x}^{(n)}$ and $R(t)$, the total angular momentum is

$$L_i = \epsilon_{ijk} \sum m^{(n)} \left[R_{jl}\tilde{x}_l^{(n)}\, R_{km}\dot{\tilde{x}}_m^{(n)} + R_{jl}\tilde{x}_l^{(n)}\, \dot{R}_{km}\tilde{x}_m^{(n)} \right] \tag{3.3}$$

To find the gauge potential $A(t) \equiv R^{-1}\dot{R}$, we set $L_i = 0$ and solve. The result, after a few lines of algebra, is

$$A(t)_{ij} = (R^{-1}\dot{R})_{ij} = \epsilon_{ijk}\tilde{I}^{-1}_{kl}\,\tilde{L}_l \tag{3.4}$$

where $\tilde{I}$ is the inertia tensor of the standardly oriented shape $S_0(t)$ and $\tilde{L}$ is the apparent angular momentum of $S_0(t)$ at time t:

$$\tilde{I}_{ij} \equiv \sum_n m^{(n)}\left((\tilde{x}^{(n)})^2\delta_{ij} - \tilde{x}^{(n)}_i\tilde{x}^{(n)}_j\right) \tag{3.5}$$

$$\tilde{L}_i \equiv \epsilon_{ijk}\sum_n m^{(n)}\,\tilde{x}^{(n)}_j\dot{\tilde{x}}^{(n)}_k \tag{3.6}$$

The formulae (2.5) and (3.4) in principle provide a complete, and rather elegant, solution to the problem of computing net rotations of a deformable body. Often, however, it is easier to compute the gauge potential directly, as we shall now do for a simple example.

4. An Example

We now consider the example of two concentric spheres rotating about their common center of mass, as depicted in Fig. 2. The space of possible orientations for each sphere is $SO(3)$ (familiarly parameterized by Euler angles), and the full configuration space of the system is $\mathcal{S} = SO(3)\times SO(3)$. The space $\mathcal{S}_0$ of standard shapes may be thought of as the space of relative orientations of the two spheres; it, too, is isomorphic to $SO(3)$. We shall choose as standard shapes those configurations in which the outer sphere is in a fixed orientation, say, with the north pole pointing in the direction of the positive z-axis and the $\phi = 0$ meridian in the xz-plane.

We can easily write down the gauge potential at an arbitrary point in $\mathcal{S}_0$ up to proportionality. Indeed, any rotation of the inner sphere relative to the outer must be compensated by an opposing rotation of the outer sphere, in order to conserve angular momentum. In the particular basis of standard shapes we have chosen, this rotation of the outer sphere is equal to the net rotation of the system. So, letting J_i be the three generators of relative rotations, the net rotation due to an infinitesimal change of shape $\Omega = \omega_i J_i$ is

$$A = -\alpha\,\Omega \tag{4.1}$$

where α is a proportionality constant between 0 and 1. Whereas the relative orientation of the two spheres at time t is

$$R_0(t) = \overline{\mathrm{P}}\exp\int^t \Omega(t')\,dt' \tag{4.2}$$

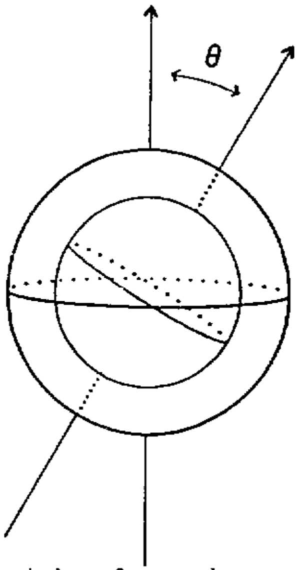

Figure 2. A body consisting of two spheres rotating about a common center of mass. Shapes are labeled by the angle θ between their polar axes.

the net rotation of the system is

$$R(t) = \overline{\mathrm{P}} \exp - \alpha \int^{t} \Omega(t')\, dt' \tag{4.3}$$

Now, even if $R_0(T) = \mathbf{1}$, there is no reason for $R(T)$ to be trivial. Thus by purely internal rearrangements, even with nothing to push against, our system can reorient itself.

We may evaluate the path ordered exponential for infinitesimal closed paths (and thus the field strength) by a simple argument. Suppose we rotate the outer sphere about the x-axis and the y-axis successively, by an angle ϵ, and then about the x-axis and the y-axis by $-\epsilon$. Finally, we close the path in shape space with a rotation about the z-axis by $-\epsilon^2$. We come back to the same shape we started at because of the general properties of rotations, as expressed by the equation

$$e^{-i\epsilon^2 J_z} e^{-i\epsilon J_y} e^{-i\epsilon J_x} e^{i\epsilon J_y} e^{i\epsilon J_x} = \mathbf{1} \tag{4.4}$$

true to order ϵ^3. According to Eq. (4.1), the net change in orientation of the inner sphere is

$$e^{i\alpha\epsilon^2 J_z} e^{i\alpha\epsilon J_y} e^{i\alpha\epsilon J_x} e^{-i\alpha\epsilon J_y} e^{-i\alpha\epsilon J_x} = e^{i(\alpha-\alpha^2)\epsilon^2 J_z} \tag{4.5}$$

The net rotation is $\epsilon^2(\alpha - \alpha^2)$ about the z–axis and the field strength F_{xy} is $(\alpha - \alpha^2)J_z$. More generally, by rotational invariance, the field strength at any point in shape space is

$$F_{ij} = (\alpha - \alpha^2)\epsilon_{ijk} J_k \tag{4.6}$$

Thus, the field strength is a monopole field [6][9] of strength $\alpha - \alpha^2$, located at the origin of the tangent plane at a point of shape space. Note that for $\alpha = 0$ or 1, F_{mn} vanishes, as it should. Furthermore, the net rotation is maximized when $\alpha = \frac{1}{2}$. An easy calculation will reveal that $\alpha = I/(I + I')$, where I and I' are the moments of inertia of the two spheres, so that $\alpha = \frac{1}{2}$ when $I = I'$.

To find α, we begin with the equation for the angular momentum:

$$L = I\dot{\theta} + I'\dot{\theta}' = 0 \tag{4.7}$$

where θ and θ' are the physical orientations of the two spheres in the plane of rotation. If θ' refers to the outer sphere, then the specification of a sequence of standard shapes means that we know $\theta - \theta'$ at all times. From this information, it is easy to obtain $\dot{\theta}'(t)$:

$$\dot{\theta}' = -\frac{I}{I + I'}(\dot{\theta} - \dot{\theta}') \tag{4.8}$$

Comparing with Eq. (4.1) gives

$$\alpha = \frac{I}{I + I'} \tag{4.9}$$

Note that α is always between 0 and 1, as claimed previously.

The foregoing example is readily generalized to more complicated situations. For instance, if we vary the moments of inertia of the spheres with time, the only modification of the above calculation that needs to be made is to bring α under the integral sign in Eq. (4.3):

$$R(t) = \overline{\mathrm{P}} \exp - \int^t \alpha(t')\Omega(t')\, dt' \tag{4.10}$$

Even the most general case of two bodies with arbitrary time-dependent inertia tensors rotating about a common center or mass is hardly any more difficult to write down:

$$R(t) = \overline{\mathrm{P}} \exp - \int^t (I + I')^{-1} I\, \Omega(t')\, dt' \tag{4.11}$$

For two bodies whose centers of mass do not coincide, the solution is more involved. In [3], the rotation of a simple system consisting of two rods joined at a hinge is solved in closed form, with overall orientation expressed as a function of the hinge angle. That result generalizes readily to more complicated systems—we leave it to the interested reader to work out the details.

5. Remarks

As we mentioned in the introduction, there are a variety of contexts where one might wish to find the net rotation of a self-deforming body. The diverse catalog of such bodies includes divers performing multiple twists [3], cats in free fall [1][2], and astronauts and satellites in space [10]. By way of example, we shall briefly consider the application of our ideas to satellites.

There are two primary means of changing the orientation of a satellite—propulsive and mechanical. The former relies on external thrusters to impart an angular momentum to the satellite, which is cancelled by a reverse thrust when the desired orientation is reached. This method has the disadvantage of requiring the initial and final thrusts to be precisely equal and opposite, a problem not shared by the second method, which falls under the general category of rotation by self-deformation. One might implement it, for example, by mounting two perpendicular flywheels near the center of the satellite. The flywheels would be used to generate rotations about the body's x- and y-axes. (A rotation about the z-axis could be generated by a sequence of x and y rotations.) The calculation of Sec. 4 may be taken over almost directly, if the flywheels are constructed so that they share a common center of mass with the rest of the satellite. Otherwise, the calculation may be modified in accord with the remarks at the end of Section 4. Note that no matter how the mechanical approach is implemented, the body's initial and final angular momentum are guaranteed to be the same.

Once the net rotation of a body due to any sequence of deformations is known, it is natural to try to optimize. Thus we ask, what is the most efficient way for a body to change its orientation? The answer will depend on many factors, including the definition of efficiency and constraints on the space of possible shapes. If one considers only infinitesimal deformations of a particular shape S_0, then the methods of [11] may be applied. However, large deformations are trickier, due to the path ordering in Eq. (2.5). The following qualitative observations bring out the subtleties of this problem. First, it is necessary to take account of non-contractible paths in shape space (*e.g.*, rotation of a wheel by 2π). Also in considering large deformations, one might expect that one could increase efficiency, by first deforming to a region of shape space with high curvature (large F_{mn}) and then performing small changes of shape. Where the curvature is large, a little bit of internal motion generates a large motion through space. Conversely, where there is small curvature even large internal motions generate only small motions through space. Hence large curvature configurations are appropriate when one wants to generate gross motions most efficiently, whereas small curvature configurations are appropriate to insure noise immunity for fine motions. This is a design principle that is intuitively evident and quantifiable once the language is understood, but might otherwise be difficult to express. Actual calculations along these lines would be illuminating.

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